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LETTER TO THE EDITOR

Proliferation assisted transport in a random environment

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Abstract
We investigate the competition between barrier slowing down and proliferation induced superdiffusion in a model of population dynamics in a random force field. Numerical results in $d = 1$ suggest that a new intermediate diffusion behaviour appears. We introduce the idea of proliferation assisted barrier crossing and give a Flory-like argument to understand qualitatively this non-trivial diffusive behaviour. A renormalization group analysis close to the critical dimension $d_c = 2$ confirms that the random force fixed point is unstable and flows towards an uncontrolled strong coupling regime.

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The presence of disorder often radically changes the statistical properties of random walks. For example, random walks in a random potential are trapped in deep potential wells: this may lead to *subdiffusion*, i.e. the fact that the typical distance travelled by the walkers increases at a slower rate than the square-root of time [1]. A much studied model exhibiting this type of behaviour is the Sinai model, where particles diffuse in a random force field in one dimension [2–4]. In this case, the energy barriers typically grow as the square root of the distance, which leads to a logarithmically slow progression of the random walkers. There are also several mechanisms that lead to *superdiffusion*. For example, if the random force field is rotational, the random walkers can be convected far away by long streamlines [1]. Another interesting mechanism of superdiffusion is *random proliferation*: suppose that each random walker can either die or give birth to new random walkers at a rate which is random, both in time and space. There is in this case a possibility for an 'outlier' random walker, that has by chance travelled a distance much greater than the square-root of time, to have been particularly prolific: he and his siblings then represent an appreciable fraction of the whole population, leading to a motion of the centre of mass faster than normal diffusion. This mechanism has been widely studied (although not explicitly discussed as such) in the context of directed polymers (DP) in random media or equivalently the Kardar–Parisi–Zhang (KPZ) model of surface growth [5]. The aim of this letter is to investigate the case where both these mechanisms are simultaneously present. The motivations for such a mixed model are numerous. In the context of population dynamics (for example, bacteria on a random substrate), similar models have recently been investigated, with quite interesting results [6]. One can also give an economic interpretation of population dynamics, where the local density of random walkers is the wealth of a given individual. Biased diffusion represents trading between individuals, whereas the random growth term is the result of speculation [7]. One can argue that generically, this type of model leads to a Pareto (power-law) tail in the distribution of wealth [7]. Finally, from a theoretical point of view, this mixed model leads to the interesting possibility of new behaviour, intermediate between superdiffusion and subdiffusion.

More precisely, we study here the following equation for the local population density $P(\vec{x}, t)$ in d dimensions (in the Stratonovich sense)

$$\frac{\partial P(\vec{x}, t)}{\partial t} = \nu_0 \Delta P(\vec{x}, t) - \vec{\nabla}(\vec{F}(\vec{x})P) + \eta(\vec{x}, t)P(\vec{x}, t) \tag{1}$$

where ν_0 is the bare diffusion constant, $\vec{F}(\vec{x})$ a space dependent static Gaussian random force such that[4] $\langle F_\mu(\vec{x})F_\nu(\vec{x}')\rangle_F = 2\sigma_F^2 \delta_{\mu,\nu}\delta^d(\vec{x} - \vec{x}')$, and $\eta(\vec{x}, t)$ a Gaussian random growth rate, depending both on space and time, with $\langle \eta(\vec{x}, t)\eta(\vec{x}', t')\rangle_\eta = 2\sigma_\eta^2 \delta(t - t')\delta^d(\vec{x} - \vec{x}')$. The initial condition is chosen to be $P(\vec{x}, t = 0) = \delta^d(\vec{x})$. Due to the last term, the total population $Z(t) = \int \mathrm{d}\vec{x}\, P(\vec{x}, t)$ is not conserved. The quantities of interest, which describe how the population spreads in time are, for example, the average centre of mass motion,

$$x_{\mathrm{cm}}^2(t) = \left\langle \left(\frac{1}{Z} \int \vec{x} P(\vec{x}, t)\, \mathrm{d}\vec{x} \right)^2 \right\rangle_{F,\eta} \tag{2}$$

or the average 'width' of the diffusing packet $\langle \Delta^2 \rangle_{F,\eta}$:

$$\Delta^2(t) = \frac{1}{Z} \int \vec{x}^2 P(\vec{x}, t)\, \mathrm{d}\vec{x} - \left(\frac{1}{Z} \int \vec{x} P(\vec{x}, t)\, \mathrm{d}\vec{x} \right)^2 . \tag{3}$$

(Other moments can, however, also be studied: see later.) An alternative description is in terms of the free-energy $h(\vec{x}, t) = \log P(\vec{x}, t)$, which obeys the equation

$$\frac{\partial h(\vec{x}, t)}{\partial t} = \nu_0 \Delta h(\vec{x}, t) + \lambda(\vec{\nabla} h)^2 - \vec{F}(\vec{x}) \cdot \vec{\nabla} h - \vec{\nabla} \cdot \vec{F}(\vec{x}) + \eta(\vec{x}, t) \tag{4}$$

with $\lambda = \nu_0$. When $\vec{F} \equiv 0$, these equations represent the well known KPZ (or directed polymer) problem, whereas for $\eta \equiv 0$, one recovers the problem of a random walk in a random environment. Both problems can be approached using a perturbative renormalization group; interestingly, the critical dimension for both problems is $d_c = 2$. For the random drift problem, one finds that the coupling constant $g_F = \sigma_F^2/(2\pi)\nu_0^2$ flows towards a non-trivial fixed point of order ϵ in dimensions $d = 2 - \epsilon$ [1, 8, 9]. This in turn leads to subdiffusive behaviour: $x_{\mathrm{cm}}(t)$ grows as t^{ν_F} with $\nu_F = (1 - \epsilon^2)/2 < 1/2$. For the KPZ problem, the coupling constant is $g_\eta = \sigma_\eta^2 \lambda^2/(2\pi)\nu_0^3$; the Gaussian fixed point $g_\eta = 0$ is again unstable for $d < 2$, but there is no accessible fixed point at one loop for $d > 3/2$ [10]. The exponent ν, therefore, cannot be computed but is expected (and found numerically) to be greater than $1/2$: in population dynamics language, the possibility of far-away proliferation leads to superdiffusion. We have performed an RG analysis in the *mixed* case where both g_F and g_η are non-zero. This can be done using a field theoretical representation [10] representation of equation (4), which allows

[4] Note that for this choice of correlator, the force is the derivative of a random potential only in $d = 1$.

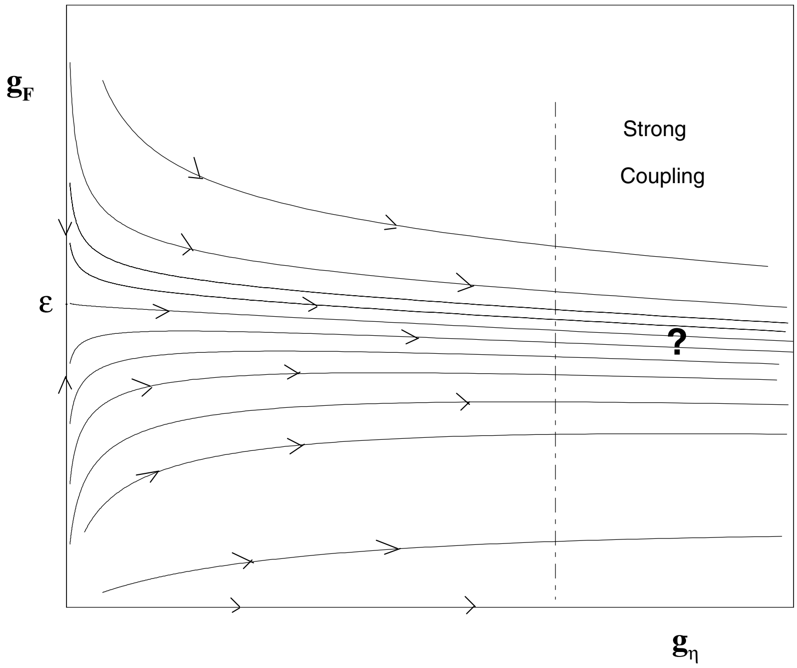

Figure 1. One-loop RG flow in the g_η, g_F plane. As soon as g_η is non-zero, it flows towards the strong coupling region. Our numerical simulations suggest that an attractive 'mixed' fixed point appears in this region, differing from the KPZ fixed point.

one to generate the perturbation expansion in g_F and g_η. Performing calculations along the lines of [8–10], we find that the two β functions are given by

$$\begin{aligned}\frac{\mathrm{d}g_\eta}{\mathrm{d}\ell} &= \epsilon g_\eta + 2g_\eta g_F + \frac{g_\eta^2}{4} \\ \frac{\mathrm{d}g_F}{\mathrm{d}\ell} &= \epsilon g_F - \frac{\epsilon g_\eta g_F}{4} - g_F^2\end{aligned} \tag{5}$$

where ℓ is the logarithm of the running length scale. The resulting flow is represented in figure 1. The subdiffusive random force fixed point $g_\eta = 0$, $g_F = \epsilon$ is therefore unstable in the presence of a small 'proliferation' term g_η. Unfortunately, at one loop, g_η flows towards the strong coupling region, as is the case in the standard KPZ case $g_F = 0$. One can, however, argue, using a path integral representation (see later), that the random force term is a relevant perturbation near the strong KPZ fixed point. Therefore, we expect the mixed problem $g_F \neq 0$, $g_\eta \neq 0$ to be described by a new strong coupling fixed point. Note that the present problem is similar to the KPZ problem with columnar noise [11], although with a special kind of correlations of the disorder.

In order to obtain some information about this strong coupling behaviour, we have performed some numerical simulations in one dimension, where both the fixed points corresponding to Sinai subdiffusion and to DP/KPZ superdiffusion are well understood. We have found results that suggest the existence of an *attractive* fixed point, characterized by a new non-trivial diffusive behaviour (intermediate between the Sinai and DP/KPZ behaviour). We have numerically evolved a space and time discretized version of equation (1), and have worked with $\log P$ to avoid precision problems. Starting from a localized packet $P(x = \mathrm{i}a, t = 0) = \delta_{i,0}$, we have found that as soon as both coupling constants, g_F^0 and

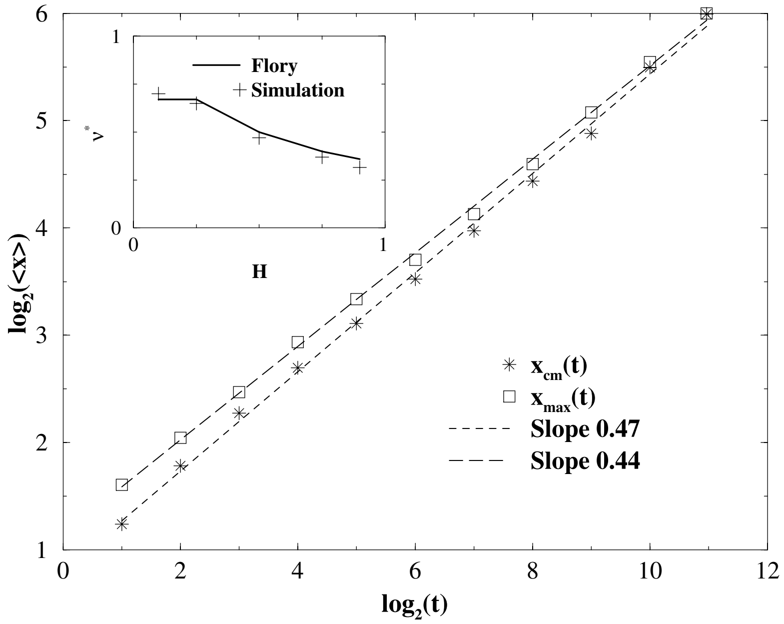

Figure 2. Behaviour of the average centre of mass $x_{\rm cm}(t)$ and of the average position of the maximum of the packet $x_{\rm max}(t)$ (i.e. the point where $P(x,t)$ is maximum), as a function of time, for $\sigma_\eta/\sigma_F = 0.125$. The best linear fits are shown, and lead to an estimate for ν^* slightly smaller than $1/2$. Inset: value of ν^* (determined from the behaviour of $x_{\rm cm}(t)$) as a function of the Hurst exponent of the potential H, compared with the Flory prediction.

g_η^0, are non-zero, the exponent ν describing the diffusion of the centre of mass $x_{\rm cm}(t)$ at large times is found to be close to the value $\nu^* = 1/2$ (see figure 2). The position $x_{\rm max}(t)$ of the maximum of $P(x,t)$ behaves very similarly. The ratio g_F^0/g_η^0 affects only the short time transient behaviour, which is either Sinai-like or DP/KPZ-like, as shown in figure 3. In the RG language, this suggests that a non-trivial *attractive* fixed point, g_F^*, g_η^*, appears. This is compatible with the flow diagram of figure 1, although this new fixed point is out of reach at the one-loop level. Although the value of $\nu^* = 1/2$ corresponds to free-diffusion, the motion of the packet for a given environment is far from a simple diffusion, as the study of the width Δ of the packet shows. We have found numerically that $\langle \Delta^q \rangle_{F,\eta}$ behaves as $t^{q\zeta_q}$ with $\zeta_1 \simeq 0.24$, $\zeta_2 \simeq 0.34$ and $\zeta_4 \simeq 0.38$. This non-trivial behaviour is actually present for both the Sinai problem and the DP/KPZ problem. This comes from the fact that, for both problems, the effective free energy $h(x,t) = \log P(x,t)$ behaves as a random walk in x space. This is trivial for the Sinai problem, since the potential is, indeed, constructed as the sum of local random forces. For the DP/KPZ problem, this is far less trivial and results from the fact that one can obtain exactly the stationary distribution of $h(x,t)$ in one dimension, which turns out to be the same as for the linear case $\lambda = 0$, i.e. again a random walk in x space [5]. It is well known that for a random walk potential, the probability that two nearly degenerate minima are separated by a distance Δ falls off as $\Delta^{-3/2}$ for large Δ. The qth moment of Δ is therefore dominated by extreme events as soon as $q > 1/2$. Physically, this means that for most realizations of the disorder, the width Δ of the packet is small [12, 13], except in rare situations where the packet is divided into two subpackets very distant from one another. The natural cut-off for Δ is of the order of $x_{\rm cm}(t)$ itself. Therefore one obtains, for $q > 1/2$, $\langle \Delta^q(t) \rangle \propto [x_{\rm cm}(t)]^{q-1/2}$.

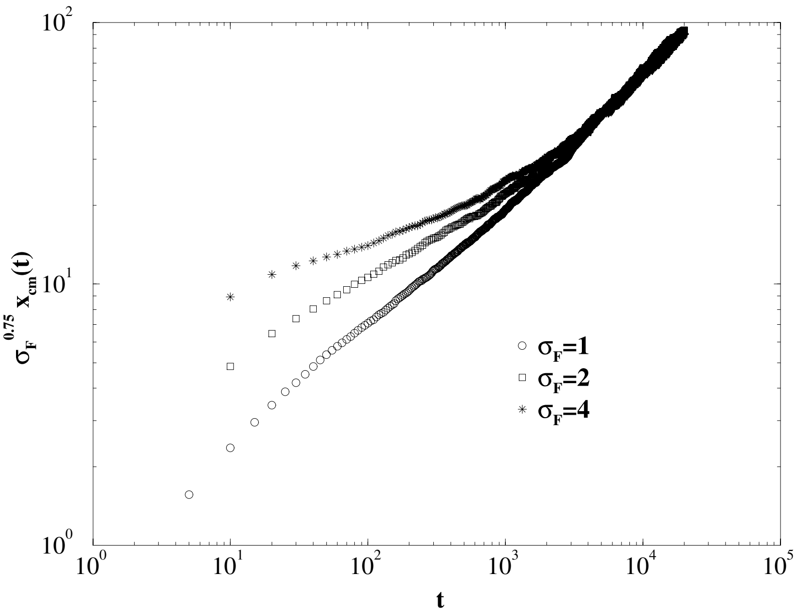

Figure 3. Behaviour of the average centre of mass $x_{\mathrm{cm}}(t)$ for different values of σ_F. This curve shows that for large σ_F, the short time behaviour is Sinai-like, crossing over to mixed behaviour at long times. The value of $x_{\mathrm{cm}}(t)$ has been rescaled by $\sigma_F^{3/4}$ to obtain a reasonable data collapse at long times.

For the Sinai problem, using $x_{\mathrm{cm}}(t) \propto \log^2(t)$, this leads to $\langle \Delta^2(t)\rangle_F \propto \log^3(t)$, whereas for the DP/KPZ case, using $x_{\mathrm{cm}}(t) \propto t^{2/3}$, one finds $\langle \Delta^2(t)\rangle_\eta \propto t$: both these results are actually exact, as has been shown in [4, 13, 14]. Assuming that the effective potential in the mixed case is again a random walk in x space, and using $\nu^* \simeq 1/2$, we obtain $\zeta_q = (2q-1)/4q$, i.e. $\zeta_1 \simeq 0.25$, $\zeta_2 \simeq 0.375$ and $\zeta_4 \simeq 0.4375$, in reasonable agreement with our numerical values[5]. In order to understand the value of $\nu^* \simeq 1/2$, one needs to develop a consistent picture of the competition between the slowing down induced by the ever-growing Sinai barriers and the speeding up of the population spreading allowed by the multiplicative growth term η. Before addressing the full Sinai + KPZ problem, we first consider the simpler case of a unique barrier of height U_0, which develops on scale L. For definiteness, we have solved numerically equation (1) on the interval $[0, L]$ with $F(x) = -U_0/L \sin(4\pi x/L)$. The initial condition is localized in the first well, and the crossing time τ is defined as the average time after which the relative weight of the population in the second well is one-half of that in the first well. For $\eta \equiv 0$, one finds the classical Arrhenius law: $\log\tau = U_0/\nu_0$. When $\eta \neq 0$, the behaviour of τ as a function of U_0 for different values of L is shown in figure 4. The result can be expressed as: $\tau \propto L^{3/2} f(U_0/\sqrt{L})$, with $f(y \to 0) = 1$ and $f(y \to \infty) \propto y^b$. This scaling of τ with L can easily be understood. In the limit $U_0 \to 0$, the time for the particles to reach a distance L is given by the DP/KPZ scaling, i.e. $L \propto \tau^{2/3}$.

The influence of the external potential U_0 becomes substantial when it becomes of the order of the effective KPZ potential h, which, as discussed above, grows as $\sqrt{L}$. Numerically, the exponent b is found to be very close to $b = 5/3$. Therefore, the exponential increase of

[5] One actually observes a similar bias towards values smaller than the theoretical one for the KPZ equation. In this case, one expects $\zeta_q = 2/3 - 1/3q$: see [13, 14].

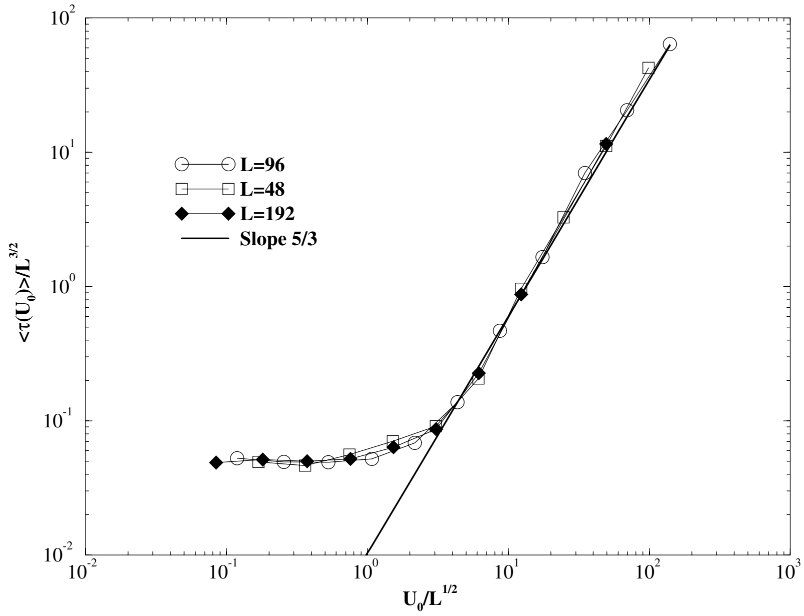

Figure 4. Average barrier crossing time $\langle\tau\rangle$, rescaled by $L^{3/2}$, as a function of the barrier height U_0 rescaled by $\sqrt{L}$, for different sizes L, and in log–log coordinates. The slope 5/3 is shown for comparison. The power law increase of $\langle\tau\rangle$ as a function of U_0 has to be compared to the usual exponential (activated) dependence.

the crossing time with the barrier height is replaced by a power-law increase in the presence of the random growth term η. One can call this effect proliferation assisted barrier crossing: the probability that a particle reaches the top of the barrier x^* by pure diffusion is $\exp(-U_0/\nu_0)$, but, due to the random growth term, this probability is multiplied by a certain proliferation 'gain' factor[6] $\exp \mathcal{G}(x^*, t)$. This factor can be estimated using a path integral representation of the population density

$$P(x,t) = \int_{(0,0)}^{(x,t)} \mathrm{d}\mathcal{C} \exp\left(-\int_0^t \mathrm{d}t' \left[\frac{1}{2\nu_0}\left(\partial_{t'}x(t') - F(x(t'))\right)^2 - \eta(x(t'),t')\right]\right) \tag{6}$$

where $F(x)$ has the previously defined sinusoidal shape. The force term gives rise to the detrimental contribution $\exp(-U_0/\nu_0)$, while the proliferation term originates the gain factor $\mathcal{G}$. If the path $\mathcal{C}$ leading from the initial point x_0 to x^* was unique, one would simply have $\mathcal{G}(x^*, t) = \int \mathrm{d}t'\, \eta(x_{\mathcal{C}}(t'), t')$, which typically behaves as $\sigma_\eta\sqrt{t}$. In fact, many paths contribute to $\mathcal{G}(x^*, t)$. This leads to a kind of pre-averaging effect of the random growth term η over the width $w(t)$ of the paths $\mathcal{C}$. Therefore[7]

$$\mathcal{G}(x^*, t) \sim \sigma_\eta \left(\int_0^t \frac{\mathrm{d}t'}{w(t')^d}\right)^{1/2}. \tag{7}$$

[6] Note that this proliferation assisted barrier crossing can only take place if the population density $P(x, t)$ can take exponentially small values. This would not be the case, for example, if one simulates equation (1) using a large but finite number of particles.

[7] A similar argument can be made to show that in the completely random problem the static random force term is a relevant perturbation in the vicinity of the KPZ fixed point: in the path integral representation (where $F(x)$ is now random) one can show that the typical contribution of this static random force to the free energy is always larger (at large length scales) than the typical fluctuations of the KPZ free energy.

Since most paths leading to x^* spend their time in the thermally accessible region of the well, one can estimate $w(t')$ as $w = L/\sqrt{U_0}$. The proliferation factor then compensates the barrier when $\tau \propto U_0^{3/2}$ (for $d = 1$). This simple argument, therefore, leads to $b = 3/2$, not very far from the numerical value $b \simeq 5/3$. Actually, one can apply this argument to the unconfined case $U_0 = 0$, where the detrimental factor is now the entropy of the random walk $\exp(-x^{*2}/t)$. Using self-consistently $w(t') = x^*(t')$, the compensation argument now leads to $x^* \propto t^{3/(4+d)}$, which is precisely the Flory result for the DP/KPZ problem. This Flory value can be obtained using a variational method, either with replicas [15] or without replicas [16]. In spirit, equation (7) is actually very close to the latter calculation. The value $b = 3/2$ can, therefore, be seen as a Flory value for this problem.

Returning now to the Sinai case, where the barrier height grows as $\sigma_F\sqrt{x^*}$, the self-consistent compensation argument now leads to $\sigma_F\sqrt{x^*} \sim \sigma_\eta\sqrt{t/x^*}$, or $x^* \sim (\sigma_\eta/\sigma_F)\sqrt{t}$. The $\sqrt{t}$ behaviour is close to the numerical results shown in figure 2. However, as shown in figure 3, the dependence of $x_{\rm cm}$ on σ_F is found to be weaker than the $1/\sigma_F$ behaviour predicted by this simple argument, and closer to $1/\sigma_F^{3/4}$. We have also investigated numerically the case where the force derives from a fractional Brownian motion with a Hurst exponent H. The case $H = 1/2$ is the standard Sinai random walk potential considered above. An extension of the proliferation argument to this case predicts that $\nu^* = 1/(1+2H)$ for $H > 1/4$, reverting to the DP/KPZ value $\nu^* = 2/3$ for smaller values of H (i.e. when the potential is not 'confining' enough). As shown in figure 2, our numerical values for ν^* agree quite well with this prediction: for example $\nu^*(H = 3/4) \simeq 0.37$ and $\nu^*(H = 1/4) \simeq 0.65$.

In summary, we have investigated the competition between barrier slowing down and proliferation induced superdiffusion in a model of population dynamics in a random force field. The one-loop RG analysis close to the critical dimension $d_{\rm c} = 2$ predicts that the subdiffusive fixed point is unstable against 'proliferation' and flows to strong coupling. Our numerical results in $d = 1$ actually suggest that both the Sinai and KPZ fixed points are unstable and flow towards a new stable mixed fixed point. We have given a heuristic Flory-like argument, which allows us to understand qualitatively the diffusive behaviour at this mixed fixed point, and also our results on proliferation assisted barrier crossing. This work can be extended in various directions: for example a two-loop RG calculation would be interesting. One could also study the effect of nonlinear terms in the population equation, such as $-P^2$ or $\vec{\nabla}\cdot(P\vec{\nabla}P)$, and the role of a non-zero external force $\langle F(x)\rangle$. It would be worth performing some numerical simulations of the barrier crossing problem and of the mixed model in $d = 2$.

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